INDIAN STATISTICAL INSTITUTE
Mid-Semestral Exam
Algebra-II
2019-2020

Total marks: 100
Time: 3 hours

Answer Question 1 and any 4 from the rest.

(1) State whether the following statements are true or false. Justify your answers.
(a) Any four vectors in $\mathbb{R}^3$ are linearly dependent.
(b) The linear system of equations $Ax = b$ is consistent if and only if the ranks of the matrix A and the augmented matrix $(A|b)$ are the same.
(c) The set $S = \{1, sin(x), cos(x)\}$ is a linearly independent subset of the set of all continuous functions from $[-\pi, \pi]$ to $\mathbb{R}$ over $\mathbb{R}$.
(d) The set $\{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 0\}$ forms a subspace of $M_{n\times n}(\mathbb{R})$. (5 × 4)

(2) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

as an element of $\mathbb{R}^{3\times 5}$.
(a) Find a basis for the null space of A.
(b) Repeat (a) when A is considered to be a matrix over $\mathbb{F}_3$. (10 +10)

(3) (a) Prove that if S is any linearly independent set in a vector space V, then S can be extended to get a basis of V.
(b) Find a basis of $\mathbb{R}^3$ containing the vectors $(1, 1, -2)^t$ and $(1, 2, -1)^t$.
(c) Find the total number of subspaces of $(\mathbb{F}_2)^2$. Explain your answer. (8+6+6)

(4) (a) Define *rank* of a matrix.
(b) State and prove *Rank-Nullity Theorem.*
(c) Show that if A is a invertible, then $\text{rank}(AB) = \text{rank}(B)$. (2 + 10 +8)

(5) (a) Prove that the set $\mathcal{B} = \{(1, 2, 0)^t, (2, 1, 2)^t, (3, 1, 1)^t\}$ is a basis of $\mathbb{R}^3$.
(b) Find the co-ordinate vector of $v = (1, 2, 3)^t$ with respect to the basis $\mathcal{B}$.
(c) Let $\mathcal{B}' = \{(0, 1, 0)^t, (1, 0, 1)^t, (2, 1, 0)^t$ be another basis of $\mathbb{R}^3$. Find the change of basis matrix relating $\mathcal{B}$ and $\mathcal{B}'$. (4+6+10)

(6) Let $P_3(\mathbb{R})$ be the space of all real polynomials of degree less than or equal to 3. Consider the ordered basis $\mathcal{B} = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$. Define the linear operator $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ by $T(1) = 1, T(x) = 1 + x, T(x^2) = (1 + x)^2, T(x^3) = (1 + x)^3$ on the basis elements.
(a) Find the matrix of T with respect to the basis $\mathcal{B}$.
(b) Prove that T is invertible.
(c) Find the matrix of T^{-1} with respect to the basis $\mathcal{B}$. (8+4+8)